Worlds:
Modal Logic vs. Two-Sorted Type Theory

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Here we will present and briefly compare two formal languages used to capture intensionality in natural language (NatLg): Modal Predicate Logic (ModPrL) –without world terms– and Two-Sorted Type Theory (Ty2) –with world terms and quantification over them.

1. Modal Predicate Logic (ModPrL)

A simple example of intensional language is Modal Predicate Logic. ModPrL is like standard Predicate Logic with the addition of the syncategorematic symbols $\Box$ and $\Diamond$. For a model M of the sort described in (1) and an assignment function g from variables to individuals in M, the interpretation of terms and formulas is defined in (2) and (3) (see Gamut, 1991: vol. II, ch 3.).

(1) A model M for a modal predicate logic language L consists of:
- i. a non empty set of possible worlds W
- ii. an accessibility relation R on W [1]
- iii. a domain function assigning a domain of individuals D_w to each $w \in W$
- iv. an interpretation function I such that:
  - a. I assigns an entity I(c) to each constant c of L, and
  - b. for every world $w \in W$, I assigns a subset $I_w(P)$ of $(D_w)^n$ to each n-ary predicate letter P of L.

(2) Interpretation of terms:[2] $[\![t]\!]^{M,w,g}$
1. If t is a constant,
   $[\![t]\!]^{M,w,g} = I(t)$ if $I(t) \in D_w$ (and undefined otherwise).
2. If t is a variable,
   $[\![t]\!]^{M,w,g} = g(t)$ if $g(t) \in D_w$ (and undefined otherwise).

[1] R can stand for the epistemic (*Epi*, e.g. with *know*), deontic (*Deo*, e.g. with *have to*), doxastic (*Dox*, e.g. with *believe*), etc. accessibility relation. E.g., $wEpiw'$ is read as 'w' is epistemically accessible from w', that is, 'for all we know in the actual world w, w' could equal w'.

[2] Here we slightly deviate from Gamut (1991).

(3) Interpretation of formulas: $[\![\phi]\!]^{M,w,g}$

3. $[\![P(t_1, ..., t_n)]\!]^{M,w,g} = 1$ iff $\langle [\![t_1]\!]^{M,w,g}, ..., [\![t_n]\!]^{M,w,g} \rangle \in [\![P]\!]^{M,w,g}$

4. $[\![\neg\phi]\!]^{M,w,g} = 1$ iff $[\![\phi]\!]^{M,w,g} = 0$

5.1. $[\![\phi \wedge \psi]\!]^{M,w,g} = 1$ iff $[\![\phi]\!]^{M,w,g} = [\![\psi]\!]^{M,w,g} = 1$

5.2. ...

6.1. $[\![\forall x[\phi]]\!]^{M,w,g} = 1$ iff for all $d \in \mathrm{D}_w$: $[\![\phi]\!]^{M,w,g^{d/x}} = 1$

6.2. $[\![\exists x[\phi]]\!]^{M,w,g} = 1$ iff for some $d \in \mathrm{D}_w$: $[\![\phi]\!]^{M,w,g^{d/x}} = 1$

7.1. $[\![\Box[\phi]]\!]^{M,w,g} = 1$ iff for all w' such that $w\mathrm{R}w'$: $[\![\phi]\!]^{M,w',g}$=1

7.2. $[\![\Diamond[\phi]]\!]^{M,w,g} = 1$ iff for some w' such that $w\mathrm{R}w'$: $[\![\phi]\!]^{M,w',g} = 1$

By way of example, the sentence (4) is translated into ModPrL as in (5) for its reading 'It must be the case that someone or other from New York won the lottery'.

(4) Someone from NY must have won the lottery.

(5) $\Box[\exists x[\mathsf{person}(x) \wedge \mathsf{from}(x, ny) \wedge \mathsf{win}(x)]$

2. Two-Sorted Type Theory (Ty2)

An alternative to intensional languages like ModPrL is the so-called Two-Sorted Type Theory (Ty2) (Gallin, 1975:58-63). Ty2 is characterized by having two sorts of basic individuals rather than one: besides regular individuals (type e), we add possible worlds as a basic type (type s). This means that a Ty2 (first-order) formal language will include variables over worlds, and that these world variables will serve as arguments of predicates and will combine with the syncategorematic symbols $\forall$ and $\exists$. The interpretation is defined in (7)-(8). A Ty2 version of ModPrL assigns the translation (9) to the aforementioned reading of (4) –with *must* interpreted at world w_0–, with direct quantification over world variables.

(6) A model M for the Ty2 version of modal predicate logic language L consists of:

i. a non-empty set of individuals D_e,

ii. a non-empty set of possible worlds D_s

iii. an accesibility relation R on D_s

iv. an interpretation function I such that:

a. I assigns an entity I(c) to each constant c of L, and

b. I assigns a subset I(P) of $\mathrm{D}_{e,1} \times ... \times \mathrm{D}_{e,n} \times \mathrm{D}_s$ to each n-ary predicate letter P of L.

(7) Interpretation of terms: $[\![t]\!]^{M,g}$

1. If t is a constant, $[\![t]\!]^{M,g} = \mathrm{I}(t)$.
2. If t is a variable, $[\![t]\!]^{M,g} = \mathrm{g}(t)$.

(8) Interpretation of formulas: $[\![\phi]\!]^{M,g}$

3. $[\![P(t_1, ..., t_n, w)]\!]^{M,g} = \mathsf{1}$ iff $\langle [\![t_1]\!]^{M,g},...,[\![t_n]\!]^{M,g}, [\![w]\!]^{M,g}\rangle \in [\![P]\!]^{M,g}$
4. $[\![\neg\phi]\!]^{M,g} = \mathsf{1}$ iff $[\![\phi]\!]^{M,g} = \mathsf{0}$

5.1. $[\![\phi \wedge \psi]\!]^{M,g} = \mathsf{1}$ iff $[\![\phi]\!]^{M,g} = [\![\psi]\!]^{M,g} = \mathsf{1}$

5.2. ...

For any variable v of type σ, where σ equals e or s:

6.1. $[\![\ \forall v_\sigma[\phi]\ ]\!]^{M,g} = \mathsf{1}$ iff for all $d \in \mathrm{D}_\sigma$: $[\![\phi]\!]^{M,g^{d/v}} = \mathsf{1}$

6.2. $[\![\ \exists v_\sigma[\phi]\ ]\!]^{M,g} = \mathsf{1}$ iff for some $d \in \mathrm{D}_\sigma$: $[\![\phi]\!]^{M,g^{d/v}} = \mathsf{1}$

(9) $\forall w'\ [\ w_0 R w' \rightarrow \exists x[\mathsf{person}(x, w') \wedge \mathsf{from}(x, ny, w') \wedge \mathsf{win}(x, w')]\ ]$

3. Comparing the two formal languages

A Ty2 language has more expressive power than ModPrL, as the former, but not the latter, will allow for a predicate within the scope of a modal to be evaluated at a higher world w. For e.g. (4), Ty2 will allow for the truth conditions in (10) –where the complex predicate *-one from NY* is in the scope of the modal but evaluated at the actual world w_0–, but standard ModPrL will not generate this reading.

(10) $\forall w'\ [\ w_0 R w' \rightarrow \exists x[\mathsf{person}(x, w_0) \wedge \mathsf{from}(x, ny, w_0) \wedge \mathsf{win}(x, w')]\ ]$

This means that the scope and the situation binding of a predicate (e.g. a noun, adjective, verb, preposition, etc.) are two different issues in Ty2 but necessarily go together in standard ModPrL: in the former, a predicate within the scope of several intensional operators can be evaluated at the world w introduced by any of these operators; in the latter, it will necessarily be evaluated at the world introduced by the closest of these operators. The crucial question is, then, whether scope and situation binding of a predicate go together in natural language –as in ModPrL– or not –as in Ty2.

4. The expressive power needed for NatLg

The empirical observation for natural language is that not all the predicates under the scope of a given intensional operator (e.g. *must*, *if*, *to say*) are evaluated at the w' introduced by that operator. For example, in the most salient reading of (11), *every* and thus its restrictor *poor child* scope under *if*, but *poor child* still denotes the set of actual poor children rather than the set of poor children in the hypothetical world.[3] In a similar fashion, in (12), *every* scopes under the second *if* but its restrictor *poor child in the neighborhood* is evaluated at the world introduced by *say*.

(11) If every poor child was rich instead, this would be a happy world.

(12) I know there aren't any poor children in this neighborhood. But, if there were poor children in this neighborhood, people would say that if every poor child in the neighborhood was rich instead, the mayor would be re-elected.

As Cresswell shows (Cresswell ,1990:ch 4), in order to capture all the possible world binding readings of natural language in ModPrL, we would need a switch operator $\mathbf{actually}_n$ for every world binder in the sentence. But a ModPrL formal language with such switch operators has the expressive power of explicit quantification over world variables, that is, the power of a Ty2 language. Ty2 gives us the translation of (11) in (13).

(13) $$\forall w \; [\; w_0 R w \;\wedge \forall x[\text{poor-child}(x, w_0) \rightarrow \text{rich}(x, w)] \\ \rightarrow \text{happy-world}(\text{this}, w) \;]$$

Hence, following Gallin (1975) and Cresswell (1990), we use Ty2 when dealing with intensionality in NatLg.

References

Gallin, D.: 1975, Intensional and Higher-Order Modal Logic with Applications to Montague Semantics, North Holland mathematics studies 19. Amsterdam: North-Holland Publ. Co.

Gamut, L.T.F.: 1991, Logic Language and Meaning. University of Chicago Press.

Cresswell, J. M.: 1990, Entities and Indices. Dordrecht: Kluwer.

[3] If *poor child* in (11) was evaluated in the hypothetical world w' introduced by *if*, w' would have the contradictory property of being such that all the children that are poor in w' are rich in w'. This is clearly not the intended reading.