Predicate Logic

- Besides keeping the connectives from PL, Predicate Logic (PrL) decomposes simple statements into smaller parts: predicates, terms and quantifiers.

(1) John is tall.
$T(j)$

(2) John visits Bill.
$V(j,b)$

(3) Everybody sleeps.
$\forall x\ [S(x)]$

(4) Somebody likes David.
$\exists x\ [L(x,d)]$

1. Syntax of PrL.

- Primitive vocabulary:

(5) Categorematic symbols (lexical entries with a denotation of their own):
a. A set of individual constants, represented with the letters **a**, **b**, **c**, **d**…
b. A set of individual variables $\mathbf{x_0}$, $\mathbf{x_1}$, $\mathbf{x_2}$,… $\mathbf{y_0}$, $\mathbf{y_1}$, $\mathbf{y_2}$,… Individual constants and individual variables together constitute the set of terms.
c. A set of predicates, each with a fixed n-arity, represented by **P**, **Q**, **R** …

(6) Syncategorematic symbols:
a. The PL logical connectives.
b. The quantifier symbols $\exists$ and $\forall$.

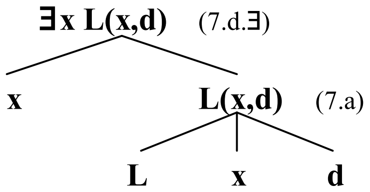
- Syntactic rules:

(7) a. If P is a n-ary predicate and $t_1 \ldots t_n$ are all terms, then $P(t_1 \ldots t_n)$ is an atomic formula.
b. If ϕ is a formula, then $\neg\phi$ is a formula.
c. If ϕ and ψ are formulae, then $(\phi \wedge \psi)$ $(\phi \vee \psi)$ $(\phi \rightarrow \psi)$ $(\phi \leftrightarrow \psi)$ are formulae too.
d. If ϕ is a formula and v is a variable, then $\forall v\phi$ $\exists v\phi$ are formulae too.
e. Nothing else is a formula in PrL.

(8) Scope and related syntactic notions:
a. If x is a variable and ϕ is a formula to which a quantifier has been attached by rule (7.d) to produce $\forall x\phi$ or $\exists x\phi$, then we say that ϕ is the *scope* of the attached quantifier and that ϕ or any part of ϕ *lies in the scope* of that quantifier.

b. An occurrence of a variable x is *bound* if it occurs in the scope of ∀x or ∃x. A variable is *free* if it is not bound. A variable is bound at most once.
c. Formulae with no free variables are called closed formulae or sentences. Formulae containing a free variable are called open formulae.

(9)



QUESTION 1: Translate into PrL the following English sentences: [From GAMUT]

(10) a. John likes Susan.

b. Everything is bitter or sweet.

c. There is something that everybody told Mary.

d. Everybody told Mary something.

e. Nobody came.

QUESTION 2: Are the following formulae sentences? In each case, which quantifier binds which variable?

(11) a. $\forall x(\exists x(R(x)\rightarrow Q(x)))$
b. $\forall x(\exists z(R(z)\rightarrow Q(x)))$
c. $\forall x(R(x,x))$
d. $\exists x(A(x,y) \wedge B(x))$
e. $\forall x\forall y A(y,y) \rightarrow B(x)$

2. Semantics of PrL.

(12) Variable assignments:
a. A variable assignment g is a function that assigns to each variable an invidual of the Universe as denotation. E.g.:

$$\begin{bmatrix} \mathbf{x} & \rightarrow & \text{Anna} \\ \mathbf{y} & \rightarrow & \text{Hans} \\ \mathbf{z} & \rightarrow & \text{Maria} \end{bmatrix}$$

b. The interpretation function, thus, does not only depend on the world of evaluation w, but also on which assignment function we use: $[\![\ ]\!]^{w,g}$

(13) a. If α is an individaul constant, then $[\![\alpha]\!]^{w,g}$ is an individual from the universe U.
b. If α is a predicate, then $[\![\alpha]\!]^{w,g}$ is a set of n-tuples over U.
c. If α is a variable, then $[\![\alpha]\!]^{w,g} = g(\alpha)$

(14) $g^{d/v}$ reads as "the variable assignment g' that is exactly like g except (maybe) for g(v), which equals the individual d".

(15) QUESTION 3: Complete the equivalences:

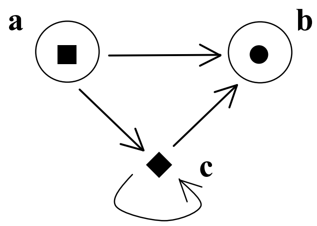
$g(\mathbf{x})$ = Mary	$g^{Paul/\mathbf{x}}$ $(\mathbf{x})$=	$g^{Paul/\mathbf{x}\ Ann/\mathbf{y}}$ $(\mathbf{x})$=
$g(\mathbf{y})$ = Susan	$g^{Paul/\mathbf{x}}$ $(\mathbf{y})$=	$g^{Paul/\mathbf{x}\ Ann/\mathbf{y}}$ $(\mathbf{y})$=

(16) a. If P is a n-ary predicate and $t_1 \ldots t_n$ are all terms, then, for any w,
$[\![P(t_1\ldots t_n)]\!]^{w,g} = 1$ iff $< [\![t_1]\!]^{w,g},\ldots,[\![t_n]\!]^{w,g} > \in [\![P]\!]^{w,g}$
If ϕ and ψ are formulae, then, for any world w,
b. $[\![\neg\phi]\!]^{w,g} = 1$ iff $[\![\phi]\!]^{w,g} = 0$
c. $[\![\phi\wedge\psi]\!]^{w,g} = 1$ iff $[\![\phi]\!]^{w,g} = 1$ and $[\![\psi]\!]^{w,g} = 1$
$[\![\phi\vee\psi]\!]^{w,g} = 1$ iff $[\![\phi]\!]^{w,g} = 1$ or $[\![\psi]\!]^{w,g} = 1$
$[\![\phi\rightarrow\psi]\!]^{w,g} = 1$ iff $[\![\phi]\!]^{w,g} = 0$ or $[\![\psi]\!]^{w,g} = 1$
$[\![\phi\leftrightarrow\psi]\!]^{w,g} = 1$ iff $[\![\phi]\!]^{w,g} = [\![\psi]\!]^{w,g}$
d. If ϕ is a formula and v is a variable, then, for any world w,
$[\![\forall v\phi]\!]^{w,g} = 1$ iff $[\![\phi]\!]^{w,gd/v} = 1$ for all the $d \in D_e$.
$[\![\exists v\phi]\!]^{w,g} = 1$ iff $[\![\phi]\!]^{w,gd/v} = 1$ for some $d \in D_e$.

(17) For any formula ϕ, $[\![\phi]\!]^{w} = 1$ iff, for all assignments g, $[\![\phi]\!]^{w,g} = 1$.

QUESTION 4: Let us take the world w depicted in (18). Let us take a language PrL_1 such that: the constants **a**, **b**, and **c** denote the individuals ■, ● and ◆, respectively, the unary predicate **A** denotes the set of individuals with a circle around, and the binary predicate **R** denotes the relation encoded by the arrows (**R(x,y)** is true iff there is an arrow from x to y). [≈GAMUT]

(18)



Determine the truth value of the following formulae of PrL_1 in w, justifying it in detail.

(19) a. $\forall x\ (A(x) \rightarrow \exists y\ (R(x,y)))$
b. $\exists x\exists y\ (R(x,y) \wedge \neg R(y,x) \wedge \exists z\ (R(x,z) \wedge R(z,y)))$
c. $\exists x\exists y\exists z\exists u\ (R(z,x) \wedge R(u,y) \wedge A(z) \wedge \neg A(u))$

■ Tips for translating from Natural Language into PrL:

(20) When translating into a universally quantified formula, the connective between the formula coming from the noun and the formula coming from the verb is usually $\rightarrow$.

E.g.: Every chicken sleeps.

$\forall x[\text{CHICKEN}(x) \rightarrow \text{SLEEP}(x)]$

(22) When translating into a existentially quantified formula, the connective between the formula coming from the noun and the formula coming from the verb is usually $\wedge$.

E.g. : Some chicken sleeps.

$\exists x[\text{CHICKEN}(x) \wedge \text{SLEEP}(x)]$

QUESTION 5: Translate into PrL the following English sentences:

(24) a. All students are smart.

b. John has a cat that he spoils.

c. John sent Mary to every professor.

d. Every man loves a woman.

e. No student from Italy talked to any professor.

f. If all logicians are smart, then Alfred is smart too.

3. Some equivalences [From Partee *et al.*]

For any predicate π and formula ϕ:

(25) Law of Quantifier Negation:

$\neg\forall x\,(\pi(x)) \;\Leftrightarrow\; \exists x\,(\neg\pi(x))$

[And, by $\neg\neg\,\phi \Leftrightarrow \phi$, also:

$\forall x\,(\pi(x)) \;\Leftrightarrow\; \neg\exists x\,(\neg\pi(x))$

$\neg\forall x\,(\neg\pi(x)) \;\Leftrightarrow\; \exists x\,(\pi(x))$

$\forall x\,(\neg\pi(x)) \;\Leftrightarrow\; \neg\exists x\,(\pi(x))$]

(26) Laws of Quantifier (In)Dependence:

a. $\forall x\forall y\,(\pi(x,y)) \;\Leftrightarrow\; \forall y\forall x\,(\pi(x,y))$

b. $\exists x\exists y\,(\pi(x,y)) \;\Leftrightarrow\; \exists y\exists x\,(\pi(x,y))$

c. $\exists x\forall y\,(\pi(x,y)) \;\Rightarrow\; \forall y\exists x\,(\pi(x,y))$

(27) Laws of Quantifier Distribution:

a. $\forall x\;(\pi(x) \wedge \rho(x)) \;\Leftrightarrow\; \forall x\,(\pi(x)) \;\wedge\; \forall x\,(\rho(x))$

b. $\exists x\;(\pi(x) \vee \rho(x)) \;\Leftrightarrow\; \exists x\,(\pi(x)) \;\vee\; \exists x\,(\rho(x))$

c. $\forall x\,(\pi(x)) \;\vee\; \forall x\,(\rho(x)) \;\Rightarrow\; \forall x\;(\pi(x) \vee \rho(x))$

d. $\exists x\;(\pi(x) \wedge \rho(x)) \;\Rightarrow\; \exists x\,(\pi(x)) \;\wedge\; \exists x\,(\rho(x))$

(28) Laws of Quantifier Movement:

a. $\phi \rightarrow \forall x(\pi(x)) \;\Leftrightarrow\; \forall x\,(\phi \rightarrow \pi(x))$

provided that x is not free in ϕ.

b. $\phi \rightarrow \exists x(\pi(x)) \;\Leftrightarrow\; \exists x\,(\phi \rightarrow \pi(x))$

provided that x is not free in ϕ and that someone exists.

c. $\forall x(\pi(x)) \;\rightarrow \phi \;\Leftrightarrow\; \exists x\;(\pi(x) \;\rightarrow \phi)$

provided that x is not free in ϕ and that someone exists.

d. $\exists x(\pi(x)) \;\rightarrow \phi \;\Leftrightarrow\; \forall x\;(\pi(x) \;\rightarrow \phi)$

provided that x is not free in ϕ.